## THE BOTT FORMULA FOR TORIC VARIETIES

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ABSTRACT. The purpose of this paper is to give an explicit formula which allows one to compute the dimension of the cohomology groups of the sheaf  $\Omega^p_{\mathbf{P}}(D) = \Omega^p_{\mathbf{P}} \otimes \mathcal{O}_{\mathbf{P}}(D)$  of p-th differential forms of Zariski twisted by an ample invertible sheaf on a complete simplicial toric variety. The formula involves some combinatorial sums of integer points over all faces of the support polytope for  $\mathcal{O}_{\mathbf{P}}(D)$ . Also, we introduce a new combinatorial object, the so-called p-th Hilbert-Ehrhart polynomial, which generalizes the usual notion and behaves similar. Namely, there exists a generalization of the reciprocity law for a usual Hilbert-Ehrhart polynomial. Some applications of the Bott formula are discussed.

## 1. Introduction

One of the most important invariants in algebraic geometry deals with the computation of cohomologies of sheaves. The crucial role in this computation is played by some vanishing criteria for higher-dimensional cohomologies of certain sheaves. This vanishing then enables one to reduce the calculation of the whole cohomology group  $H^q(\mathbf{P}, \mathcal{F})$  to the vector space of global sections  $\Gamma(\mathbf{P}, \mathcal{F}) = H^0(\mathbf{P}, \mathcal{F})$ . For example, the Bott vanishing theorem [Bo] and the Serre duality theorem help to compute the dimension of  $H^q(\mathbf{P}^n, \Omega^p_{\mathbf{P}^n}(k))$  on a complex projective space  $\mathbf{P}^n = \mathbf{P}^n(\mathbf{C})$ , where  $\Omega^p_{\mathbf{P}^n}(k) := \Omega^p_{\mathbf{P}^n} \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(k)$ .

For simplicity, we denote the dimension of the q-th cohomology group of the coherent sheaf  $\mathcal{F}$  on a complete projective variety  $\mathbf{P}$  by  $h^q(\mathbf{P}, \mathcal{F}) := \dim_{\mathbf{C}} H^q(\mathbf{P}, \mathcal{F})$ .

**Theorem 1.1** (Bott formula for  $\mathbf{P}^n$  [Bo, OSS]). Let  $\mathbf{P}^n$  be a complex projective space.

1) If k = 0, then

$$h^q(\mathbf{P}^n, \Omega^p_{\mathbf{P}^n}) = \begin{cases} 1, & p = q, \\ 0, & otherwise. \end{cases}$$

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2) If  $k \neq 0$ , then

$$h^{q}(\mathbf{P}^{n}, \Omega_{\mathbf{P}^{n}}^{p}(k)) = \begin{cases} \binom{n+k-p}{k} \binom{k-1}{p}, & q = 0, k > p, \\ \binom{-k+p}{-k} \binom{-k-1}{n-p}, & q = n, k < p-n, \\ 0, & otherwise. \end{cases}$$

This theorem was also proved by I-C. Huang [Hu], who constructed an explicit basis for the cohomology groups. Theorem 2.3.2 in [Do] gives a similar result for weighted projective spaces.

The first aim of this paper is to give a generalization of the Bott formula for a complete projective toric variety. Let D be a Cartier divisor on a toric variety  $\mathbf{P}$ , and  $\Omega^p_{\mathbf{P}}(D)$  be the sheaf of p-differential forms of Zariski on  $\mathbf{P}$ . Since we have an analogy of the Bott vanishing theorem, it is possible to compute the dimension of  $H^q(\mathbf{P}, \Omega^p_{\mathbf{P}}(D))$  when D is ample. The answer will be given in terms of combinatorial sums over all faces of the support polytope  $\Delta$  for  $\mathcal{O}_{\mathbf{P}}(D)$ , playing the role of the number k. The case when  $\mathcal{O}_{\mathbf{P}}(D) \simeq \mathcal{O}_{\mathbf{P}}$  corresponding to k = 0 was extended to a toric variety in [DK] and [O1]. Also, note that for p = 0, we get the well-known result on the cohomologies  $H^q(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(D))$  of the ample invertible sheaf  $\mathcal{O}_{\mathbf{P}}(D)$ , which can be found in any introduction to the theory of toric varieties:

$$\dim_{\mathbf{C}} H^{q}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(D)) = \begin{cases} \#(\Delta \cap M), & q = 0, \\ 0, & q > 0, \end{cases}$$

where  $\#(\Delta \cap M)$  is the number of integer points in the polytope  $\Delta$  in a lattice M. Let N be a free  $\mathbf{Z}$ -module of rank n and  $M := \operatorname{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$  its dual. Denote by  $\Sigma$  a complete fan of convex polyhedral cones in  $N_{\mathbf{R}} := N \otimes \mathbf{R}$ . We associate a so-called  $toric\ variety\ \mathbf{P} = \mathbf{P}(\Sigma)$  with  $\Sigma$ , i.e. a normal complex variety containing a torus  $T_N := \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{C}^*)$  as a dense open set with an algebraic action of  $T_N$  on  $\mathbf{P}$ . For precise definitions and basic references about toric varieties see [D1], [O1] and [F2]. In section 8 we will give a brief review of another, but equivalent, approach to toric varieties which has been introduced and studied in [Au], [C] and [BC]. Throughout this paper we assume that  $\mathbf{P}$  is complete and simplicial. Here is the precise definition of the sheaf of p-differential forms of Zariski, which is the main object of our study.

**Definition 1.2.** Denote the sheaf of p-differential forms of Zariski on  $\mathbf{P}$  by  $\Omega^p_{\mathbf{P}}$ . These forms are defined by  $\Omega^p_{\mathbf{P}} := i_* \Omega^p_U$ , where  $i: U \hookrightarrow \mathbf{P}$  is the inclusion of the nonsingular locus U of  $\mathbf{P}$ . Also, denote for any invertible sheaf  $\mathcal{O}_{\mathbf{P}}(D)$  on  $\mathbf{P}$ 

$$\Omega_{\mathbf{P}}^p(D) := \Omega_{\mathbf{P}}^p \otimes \mathcal{O}_{\mathbf{P}}(D).$$

The paper is organized as follows:

In sections 2 and 3, we prove a generalization of the original Bott formula stated above. The generalization will be given in two forms: in theorem 2.14 and in theorem 3.6. The first version of the Bott formula is proved using the Ishida-Oda complex, while the second version is proved using the technique of Danilov and Khovanskiî. Both versions are formulated in terms of sums over all faces  $\Gamma$  of the support polytope  $\Delta$ .

In section 4, we compare our results with the original Bott formula.

In section 5, we introduce the notion of the p-th Hilbert-Ehrhart polynomial  $L_p(k)$  coinciding with the usual Ehrhart polynomial L(k) when p=0. Our idea is to compare the Ehrhart polynomial with a certain Hilbert polynomial corresponding to the sheaf  $\Omega^p_{\mathbf{P}}(D)$ . Then, the Serre duality provides a generalization of the well-known reciprocity law.

In section 6, we compare the two versions of the Bott formula for toric varieties and obtain non-trivial identities between integer points in simple integer polytopes. We give a simple combinatorial proof of the identities and the generalized reciprocity law.

In section 7, we apply the Bott formula to the computation of cohomologies of quasi-smooth hypersurfaces on a complete simplicial toric variety.

In section 8, we compute the dimension of the space of global sections of weight components of the sheaf  $\Omega^p_{\mathbf{P}}(\log(-K)) \otimes \mathcal{L}$ .

In section 9, we obtain a result similar to the Bott formula on the projective bundle  $\mathbf{P}(\mathcal{E})$  associated with the sheaf  $\mathcal{E} = \mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_s$ , where  $\mathcal{L}_i$  is an ample invertible sheaf on  $\mathbf{P}$ . Namely, we compute the dimension of the global sections of the sheaf  $\Omega^p_{Y/\mathbf{P}}(k)$  of relative differential forms on  $\mathbf{P}(\mathcal{E})$ .

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## 2. The generalized Bott formula (I)

Let **P** be a complete simplicial *n*-dimensional projective toric variety. Fix an invertible sheaf  $\mathcal{L} = \mathcal{O}_{\mathbf{P}}(D)$  corresponding to an ample Cartier divisor D on **P**.

It is known from the general theory of toric varieties (see, e. g., [O1]) that the cohomology group  $H^0(\mathbf{P}, \mathcal{L})$  admits an eigenspace decomposition coming from the torus action

$$H^0(\mathbf{P},\mathcal{L}) = \bigoplus_{m \in M} H^0(\mathbf{P},\mathcal{L})_m$$

and  $H^q(\mathbf{P}, \mathcal{L})$  vanishes for q > 0.

**Definition 2.1.** Denote the set of all k-dimensional cones of  $\Sigma$  by  $\Sigma(k)$  and by  $|\Sigma(k)|$  the number of k-dimensional cones of  $\Sigma$ , i.e. the cardinality of  $\Sigma(k)$ .

**Definition 2.2** ([TE]). The convex hull  $\Delta = \Delta(\mathcal{L})$  of all lattice points  $m \in M$  in  $M_{\mathbf{R}} := M \otimes \mathbf{R}$  for which  $H^0(X, \mathcal{L})_m \neq 0$  is called the *support polytope for*  $\mathcal{L}$ . Moreover,  $\Delta$  is equal to the intersection

$$\Delta = \bigcap_{\tau \in \Sigma(n)} (m_{\tau}(\mathcal{L}) + \tau^{\vee}),$$

where  $m_{\tau}(\mathcal{L})$  is the unique lattice point of M such that

$$\{m \in M : H^0(U_\tau, \mathcal{L})_m \neq 0\} = m_\tau(\mathcal{L}) + \tau^\vee \cap M$$

for any affine chart  $U_{\tau} = \operatorname{Spec} \mathbf{C}[M \cap \tau^{\vee}]$  corresponding to the cone  $\tau$ .

Remark 2.3.  $\Delta$  is a *simple* convex polytope of dimension n, i.e. only n edges of  $\Delta$  meet at each vertex, if and only if  $\mathbf{P}$  is a simplicial toric variety.

**Definition 2.4.** Each face  $\Delta_{\sigma}$  of  $\Delta$  corresponding to the cone  $\sigma \in \Sigma$  is defined as

$$\Delta_{\sigma} := \bigcap_{\substack{\tau \in \Sigma(n) \\ \tau \succ \sigma}} (m_{\tau}(\mathcal{L}) + \tau^{\vee} \cap \sigma^{\perp}),$$

where  $\tau \succ \sigma$  means that  $\sigma$  is a face of  $\tau$ .

**Definition 2.5.** Denote by  $l(\Delta)$  the number of integer points in the polytope  $\Delta$ , and let  $l(\Delta_{\sigma})$  be the number of integer points contained in the face  $\Delta_{\sigma}$  of  $\Delta$ .

**Definition 2.6.** We identify the closure  $V(\sigma)$  of any torus-invariant orbit  $orb(\sigma)$  in **P** corresponding to  $\sigma \in \Sigma$  with a toric variety with respect to a fan glued from the cones  $(\tau + (-\sigma))/\mathbf{R}\sigma$  in  $N_{\mathbf{R}}/\mathbf{R}\tau$ ,  $\tau \in \operatorname{Star}_{\sigma}(\Sigma)$ , where  $\mathbf{R}\sigma := \sigma + (-\sigma)$  is the smallest **R**-subspace containing  $\sigma$  of  $N_{\mathbf{R}}$ , while  $\operatorname{Star}_{\sigma}(\Sigma) := \{\tau \in \Sigma : \tau \succ \sigma\}$  (see [O1]).

**Proposition 2.7** ([BC], Proposition 4.10). If  $\mathcal{L}$  is an ample invertible sheaf on  $\mathbf{P}$ , then one has a one-to-one correspondence between (n-k)-dimensional faces  $\Delta_{\sigma}$  of the polytope  $\Delta(\mathcal{L})$  and k-dimensional cones  $\sigma \in \Sigma$  reversing the face-relation. Moreover,  $\Delta_{\sigma}$  is the support polytope for the sheaf  $\mathcal{O}_{V(\sigma)} \otimes \mathcal{L}$ .

**Definition 2.8.** We denote by  $\mathcal{F}_k(\Delta)$  the set of all k-dimensional faces of  $\Delta$  and by  $f_k = |\mathcal{F}_k(\Delta)|$  the number of k-dimensional faces of  $\Delta$ .

Remark 2.9. It follows from the proposition above that  $\mathcal{F}_j(\Delta) \simeq \Sigma(n-j)$  and  $f_j = |\Sigma(n-j)|$  respectively.

**Definition 2.10.** Define the torus-invariant effective divisor  $D(\sigma)$  on each  $V(\sigma)$  by

$$D(\sigma) := V(\sigma) \backslash orb(\sigma).$$

Now recall the construction of the complex which has been introduced and studied by M.-N. Ishida and T. Oda in [Is], [O1] and [O2], known as *Ishida's p-th complex of*  $\mathcal{O}_{\mathbf{P}}$ -modules. Ishida's complex plays an important role in the proof of the first version of the generalized Bott formula.

**Proposition 2.11** ([O1], Corollary 3.2)). Let  $\Omega_{V(\sigma)}^p(\log D(\sigma))$  be a sheaf of meromorphic p-forms with logarithmic poles along  $D(\sigma)$ . There exists an isomorphism of  $\mathcal{O}_{V(\sigma)}$ -modules

$$\Omega^p_{V(\sigma)}(\log D(\sigma)) = \mathcal{O}_{V(\sigma)} \otimes_{\mathbf{Z}} \bigwedge^p (M \cap \sigma^{\perp}) \quad \text{for } 0 \leq p \leq \dim V(\sigma).$$

Now, for each pair of integers p and q we let

$$\mathcal{K}^{p,\,q}_{\mathbf{P}} := \bigoplus_{\sigma \in \Sigma(q)} \Omega^{p-q}_{V(\sigma)}(\log D(\sigma)) = \bigoplus_{\sigma \in \Sigma(q)} \mathcal{O}_{V(\sigma)} \otimes_{\mathbf{Z}} \bigwedge^{p-q} (M \cap \sigma^{\perp}) \qquad \text{if} \quad 0 \leq q \leq p,$$

and  $\mathcal{K}^{p, q}_{\mathbf{p}} = 0$  otherwise.

**Definition 2.12.** The coboundary map  $\delta: \mathcal{K}^{p, q}_{\mathbf{P}} \to \mathcal{K}^{p, q+1}_{\mathbf{P}}$  is defined to be the direct sum  $\delta = \bigoplus_{\sigma, \tau} R_{\sigma/\tau}$  of the maps

$$R_{\sigma/\tau}: \Omega^{p-q}_{V(\sigma)}(\log D(\sigma)) \to \Omega^{p-q-1}_{V(\tau)}(\log D(\tau)),$$

i.e.

$$R_{\sigma/\tau}:\,\mathcal{O}_{V(\sigma)}\otimes_{\mathbf{Z}}\bigwedge\nolimits^{p-q}(M\cap\sigma^{\perp})\to\mathcal{O}_{V(\tau)}\otimes_{\mathbf{Z}}\bigwedge\nolimits^{p-q-1}(M\cap\tau^{\perp}),$$

where  $R_{\sigma/\tau}$  is the tensor product of the restriction map  $\mathcal{O}_{V(\sigma)} \to \mathcal{O}_{V(\tau)}$  with the interior product  $\delta_{\sigma/\tau}$  whenever  $\sigma$  is a face of  $\tau$ , and zero otherwise. The definition of  $\delta_{\sigma/\tau}$  is the following. The homomorphism  $\delta_{\sigma/\tau}$  is defined to be zero when  $\sigma$  is not a face of  $\tau$ . On the other hand, if  $\sigma$  is a face of  $\tau$ , then we can determine a primitive element  $n \in N$  uniquely modulo  $N \cap \mathbf{R}\sigma$  so that  $\tau + (-\sigma) = \mathbf{R}_{\geq 0}n + \mathbf{R}\sigma$ . Moreover,  $M \cap \tau^{\perp}$  is a **Z**-submodule of corank one in  $M \cap \sigma^{\perp}$ . Each element of  $\Lambda^{p-q}(M \cap \sigma^{\perp})$  can be written as a finite linear combination of

$$m_1 \wedge m_2 \wedge \cdots \wedge m_{p-q}, \quad m_1 \in M \cap \sigma^{\perp}, \quad m_2, \ldots, m_{p-q} \in M \cap \tau^{\perp}.$$

Now define

$$\delta_{\sigma/\tau}: \bigwedge^{p-q} (M \cap \sigma^{\perp}) \to \bigwedge^{p-q-1} (M \cap \tau^{\perp})$$

by

$$\delta_{\sigma/\tau}(m_1 \wedge m_2 \wedge \cdots \wedge m_{p-q}) := \langle m_1, n \rangle m_2 \wedge \cdots \wedge m_{p-q}.$$

Actually,  $R_{\sigma/\tau}$  is the *Poincaré residue map* for the component  $V(\tau)$  of the divisor  $D(\sigma)$  on  $V(\sigma)$ .

**Theorem 2.13** ([O1], Theorem 3.6). If **P** is a simplicial toric variety, then for each  $0 \le p \le n$  there exists an exact sequence

(1) 
$$0 \to \Omega_{\mathbf{P}}^p \to \mathcal{K}_{\mathbf{P}}^{p,\,0} \xrightarrow{\delta} \mathcal{K}_{\mathbf{P}}^{p,\,1} \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{K}_{\mathbf{P}}^{p,\,p} \to 0$$

of  $\mathcal{O}_{\mathbf{P}}$ -modules on  $\mathbf{P}$ .

**Theorem 2.14** (The generalized Bott formula (I)). Let  $\mathbf{P}$  be a complete simplicial toric variety,  $\mathcal{O}_{\mathbf{P}}(D)$  be an invertible sheaf on  $\mathbf{P}$  corresponding to an ample divisor D, and  $\Delta$  be the support polytope for  $\mathcal{O}_{\mathbf{P}}(D)$ . Then

1)

$$h^{q}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}) = \begin{cases} \sum_{j=0}^{p} (-1)^{p-j} \binom{n-j}{p-j} f_{n-j}, & q = p, \\ 0, & q \neq p. \end{cases}$$

$$h^{q}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(D)) = \begin{cases} \sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} \sum_{F \in \mathcal{F}_{n-j}(\Delta)} l(F), & q = 0, \\ 0, & q > 0. \end{cases}$$

*Proof.* The assertion 1) is proved in the Theorem 3.11 of [O1] in the form

$$h^{p}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}) = \sum_{j=0}^{p} (-1)^{p-j} \binom{n-j}{p-j} |\Sigma(j)|,$$

and  $h^q(\mathbf{P}, \Omega_{\mathbf{P}}^p) = 0$  for  $p \neq q$ .

Let us prove 2). We write  $\mathcal{L}$  for  $\mathcal{O}_{\mathbf{P}}(D)$ , and  $\Omega^p_{\mathbf{P}} \otimes \mathcal{L}$  for  $\Omega^p_{\mathbf{P}}(D)$  for convenience. By Bott vanishing theorem (see [D1, O1])  $h^q(\mathbf{P}, \Omega^p_{\mathbf{P}} \otimes \mathcal{L}) = 0$  for q > 0. The sequence (1) remains exact after shifting by  $\mathcal{L}$ 

$$(2) 0 \to \Omega^p_{\mathbf{p}} \otimes \mathcal{L} \to \mathcal{K}^{p,0}_{\mathbf{p}} \otimes \mathcal{L} \to \mathcal{K}^{p,1}_{\mathbf{p}} \otimes \mathcal{L} \to \dots \to \mathcal{K}^{p,p}_{\mathbf{p}} \otimes \mathcal{L} \to 0.$$

Note, that  $M \cap \sigma^{\perp}$  is a free **Z**-module of rank n-j and  $\bigwedge^{p-j}(M \cap \sigma^{\perp})$  is a free **Z**-module of rank  $\binom{n-j}{p-j}$  when  $\sigma \in \Sigma(j)$ . This observation shows that

$$\mathcal{K}_{\mathbf{P}}^{p,j} \otimes \mathcal{L} \simeq \mathcal{O}_{V(\sigma)} \otimes_{\mathbf{Z}} \bigwedge^{p-j} (M \cap \sigma^{\perp}) \otimes \mathcal{L} \simeq \mathcal{O}_{V(\sigma)} \otimes \mathcal{L}^{\oplus \binom{n-j}{p-j}}.$$

Assume for the moment that  $h^q(\mathbf{P}, \mathcal{K}^{p,j}_{\mathbf{P}} \otimes \mathcal{L}) = 0$  for q > 0. Then (2) gives the exact sequence of global sections

$$0 \to H^0(\mathbf{P}, \Omega^p_{\mathbf{P}} \otimes \mathcal{L}) \to H^0(\mathbf{P}, \mathcal{K}^{p,0}_{\mathbf{P}} \otimes \mathcal{L}) \to H^0(\mathbf{P}, \mathcal{K}^{p,1}_{\mathbf{P}} \otimes \mathcal{L}) \to \dots$$
$$\dots \to H^0(\mathbf{P}, \mathcal{K}^{p,p}_{\mathbf{P}} \otimes \mathcal{L}) \to 0.$$

Since  $h^0(\mathbf{P}, \mathcal{O}_{V(\sigma)} \otimes \mathcal{L}) = l(\Delta_{\sigma})$  (see Proposition 2.7), we have

$$h^{0}(\mathbf{P}, \Omega_{\mathbf{P}}^{p} \otimes \mathcal{L}) = \sum_{j=0}^{p} (-1)^{j} h^{0}(\mathbf{P}, \mathcal{K}_{\mathbf{P}}^{p, j} \otimes \mathcal{L})$$

$$= \sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} \sum_{\sigma \in \Sigma(j)} h^{0}(\mathbf{P}, \mathcal{O}_{V(\sigma)} \otimes \mathcal{L})$$

$$= \sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} \sum_{\sigma \in \Sigma(j)} l(\Delta_{\sigma})$$

$$= \sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} \sum_{F \in \mathcal{F}_{n-j}(\Delta)} l(F).$$

It remains to show that  $h^q(\mathbf{P}, \mathcal{K}^{p,j}_{\mathbf{P}} \otimes \mathcal{L}) = 0$  for q > 0. Moreover, it is sufficient to prove the vanishing of  $h^q(\mathbf{P}, \mathcal{O}_{V(\sigma)} \otimes \mathcal{L}) = h^q(\mathbf{P}, \mathcal{L}|_{V(\sigma)})$ . As observed in [D1], Lemma 6.8.1, the restriction homomorphism

$$\Gamma(\mathbf{P}, \mathcal{L}) \to \Gamma(\mathbf{P}, \mathcal{L}|_{V(\sigma)})$$

is surjective. Hence, the sheaf  $\mathcal{L}|_{V(\sigma)}$  is generated by its global sections since  $\mathcal{L}$  is ample, and all cohomologies  $h^q(\mathbf{P}, \mathcal{L}|_{V(\sigma)})$  for q > 0 vanish by the Bott vanishing theorem.

Corollary 2.15. One has the following formulas

$$\sum_{p=0}^{n} h^{p}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}) y^{p} = \sum_{j=0}^{n} f_{j} (1 - y)^{j} y^{n-j} = \sum_{j=0}^{n} f_{j} (y - 1)^{j},$$

$$\sum_{p=0}^{n} h^{0}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(D)) y^{p} = \sum_{j=0}^{n} \Delta^{(j)} (-y)^{j} (y + 1)^{n-j}, \quad \text{if } D \text{ is ample,}$$

where 
$$\Delta^{(j)} := \sum_{F \in \mathcal{F}_{n-j}(\Delta)} l(F)$$
.

Notice that the first line of the equalities is contained in Theorem 3.11 of [O1], where the second equality follows from the Serre duality  $h^p(\mathbf{P}, \Omega_{\mathbf{P}}^p) = h^{n-p}(\mathbf{P}, \Omega_{\mathbf{P}}^{n-p})$ .

## 3. The generalized Bott formula (II)

Another approach gives the second version of the generalization of the Bott formula. Here we present two independent proves of this formula.

Suppose that  $\mathbf{P}$  is a complete simplicial n-dimensional toric variety as above. Assume that  $\mathcal{O}_{\mathbf{P}}(D)$  is an ample invertible sheaf on  $\mathbf{P}$  determining the convex polytope  $\Delta$ . Because of the one-to-one correspondence between the faces  $\Gamma = \Delta_{\sigma}$  of the polytope  $\Delta$  and cones  $\sigma$  of the fan  $\Sigma$ , we can associate a "small" toric variety, i.e. the closure of torus-invariant orbit of  $\mathbf{P}$ , with any face  $\Gamma \subseteq \Delta$ .

**Definition 3.1.** The toric k-dimensional subvariety  $\mathbf{P}_{\Gamma}$  of  $\mathbf{P}$  corresponding to the k-dimensional face  $\Gamma \subseteq \Delta$  is the closure of a n-k-dimensional orbit  $orb(\Gamma)$  in  $\mathbf{P}$ . For  $\Gamma = \Delta$  we let  $\mathbf{P}_{\Delta} = \mathbf{P}$ . Define the effective divisor  $D_{\Gamma}$  on each  $\mathbf{P}_{\Gamma}$  by

$$D_{\Gamma} := \mathbf{P}_{\Gamma} \backslash orb(\Gamma).$$

**Definition 3.2.** [DK, D2]. Denote by  $\Omega^p_{(\mathbf{P}_{\Gamma}, D_{\Gamma})}$  the sheaf of regular differential p-forms on the toric variety  $\mathbf{P}_{\Gamma}$  with zeros along  $D_{\Gamma}$ , arising from the exact sequence

$$(3) 0 \to \Omega^p_{(\mathbf{P}_{\Gamma}, D_{\Gamma})} \to \Omega^p_{\mathbf{P}_{\Gamma}} \overset{R_{\Gamma}}{\to} \bigoplus_{\Theta} \Omega^p_{\mathbf{P}_{\Theta}} \to 0,$$

where  $\Theta$  runs over all facets (faces of codimension one) of  $\Gamma$  and  $R_{\Gamma}$  is the restriction homomorphism.

**Definition 3.3.** The *Euler-Poincaré characteristic* of a coherent  $\mathcal{O}_{\mathbf{P}}$ -module  $\mathcal{F}$  on a n-dimensional complete variety  $\mathbf{P}$  over  $\mathbf{C}$  is defined to be

$$\chi(\mathbf{P}, \mathcal{F}) := \sum_{k=0}^{n} (-1)^k \dim_{\mathbf{C}} H^k(\mathbf{P}, \mathcal{F}) = \sum_{k=0}^{n} (-1)^k h^k(\mathbf{P}, \mathcal{F}).$$

**Definition 3.4.** Denote by  $l^*(\Delta)$  the number of integer points in the relative interior of the polytope  $\Delta$ . Also, let  $l^*(\Gamma)$  be the number of integer points contained in the relative interior of a face  $\Gamma$  of  $\Delta$ .

Notice the following key property of the sheaf  $\Omega^p_{(\mathbf{P}_{\Gamma}, D_{\Gamma})}$ .

**Proposition 3.5.** Let  $\mathbf{P}$  be a complete simplicial toric variety and  $\mathbf{P}_{\Gamma}$  be a closed subset of  $\mathbf{P}$  associated with the face  $\Gamma$  of the support polytope  $\Delta$  corresponding to an invertible sheaf  $\mathcal{O}_{\mathbf{P}}(D)$  on  $\mathbf{P}$ . Then

$$\chi(\mathbf{P}_{\Gamma}, \, \Omega^{p}_{(\mathbf{P}_{\Gamma}, \, D_{\Gamma})} \otimes \mathcal{O}_{\mathbf{P}}(D)) = \left\{ \begin{array}{ll} (-1)^{\dim \Gamma} \binom{\dim \Gamma}{p}, & \mathcal{O}_{\mathbf{P}}(D) \simeq \mathcal{O}_{\mathbf{P}}, \\ \\ \binom{\dim \Gamma}{p} l^{*}(\Gamma), & \mathcal{O}_{\mathbf{P}}(D) \not\simeq \mathcal{O}_{\mathbf{P}}, \, D \, \, is \, \, ample. \end{array} \right.$$

*Proof.* The statement follows immediately from the Proposition 2.10 of [DK] by taking the restriction of the sheaf  $\mathcal{L}$  to  $\mathbf{P}_{\Gamma}$ .

Here is the main result of this section.

**Theorem 3.6** (The generalized Bott formula (II)). Let  $\mathbf{P}$  be a complete simplicial toric variety,  $\mathcal{O}_{\mathbf{P}}(D)$  be an invertible sheaf on  $\mathbf{P}$  corresponding to an ample divisor D, and  $\Delta$  be the support polytope for  $\mathcal{O}_{\mathbf{P}}(D)$ . Then

1)

$$h^{q}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}) = \begin{cases} \sum_{s=p}^{n} (-1)^{s+p} \binom{s}{p} f_{s}, & q = p, \\ 0, & q \neq p. \end{cases}$$

2)

$$h^{q}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(D)) = \begin{cases} \sum_{s=p}^{n} \binom{s}{p} \sum_{\Gamma \in \mathcal{F}_{s}(\Delta)} l^{*}(\Gamma), & q = 0, \\ 0, & q > 0. \end{cases}$$

Proof. The formula

$$h^{p}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}) = (-1)^{p} \sum_{\Gamma \subset \Delta} (-1)^{\dim \Gamma} \binom{\dim \Gamma}{p} = \sum_{s=p}^{n} (-1)^{s+p} \binom{s}{p} f_{s}$$

is the assertion of Corollary 2.5 of [DK]. We prove this statement independently. Making use of the exact sequence (3), we get

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^p) = \sum_{\Gamma \subset \Delta} \chi(\mathbf{P}_{\Gamma}, \Omega_{(\mathbf{P}_{\Gamma}, D_{\Gamma})}^p).$$

Now the requested formula follows from Proposition 3.5 and the vanishing of  $h^q(\mathbf{P}, \Omega_{\mathbf{P}}^p)$  for  $p \neq q$ .

Let us prove assertion 2). The short exact sequence (3) remains exact after tensoring by  $\mathcal{L} = \mathcal{O}_{\mathbf{P}}(D)$ . As in the case above, we have

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^p \otimes \mathcal{L}) = \sum_{\Gamma \subseteq \Delta} \chi(\mathbf{P}_{\Gamma}, \Omega_{(\mathbf{P}_{\Gamma}, D_{\Gamma})}^p \otimes \mathcal{L}),$$

and from Proposition 3.5 and the Bott vanishing theorem,

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^p \otimes \mathcal{L}) = h^0(\mathbf{P}, \Omega_{\mathbf{P}}^p \otimes \mathcal{L}) = \sum_{\Gamma \subset \Delta} {\dim \Gamma \choose p} l^*(\Gamma) = \sum_{s=p}^n {s \choose p} \sum_{\Gamma \in \mathcal{F}_s(\Delta)} l^*(\Gamma).$$

We prove the second assertion of the theorem by using the description of the space of global sections of the sheaf  $\Omega^p_{\mathbf{P}} \otimes \mathcal{L}$ . We have a decomposition into a direct sum of M-homogeneous components

$$H^0(\mathbf{P}, \Omega^p_{\mathbf{P}} \otimes \mathcal{L}) = \bigoplus_{m \in \Delta \cap M} \bigwedge^p V_{\Delta}(m),$$

where  $V_{\Delta}(m)$  is the **C**-subspace in  $M_{\mathbf{C}} := M \otimes \mathbf{C}$  generated by the smallest face of  $\Delta = \Delta(\mathcal{L})$  containing m. Since the support polytope  $\Delta$  admits the natural partition  $\Delta = \coprod_{\Gamma} Int(\Gamma)$ , where  $Int(\Gamma)$  is the relative interior of the face  $\Gamma$ , we have

$$h^{0}(\mathbf{P}, \Omega_{\mathbf{P}}^{p} \otimes \mathcal{L}) = \sum_{\Gamma \subseteq \Delta} \sum_{m \in Int(\Gamma) \cap M} \binom{\dim V_{\Delta}(m)}{p} = \sum_{\Gamma \subseteq \Delta} \binom{\dim \Gamma}{p} l^{*}(\Gamma).$$

Thus we have given a different proof for the second formula.

Corollary 3.7. One has the following equality

$$\sum_{p=0}^{n} h^{0}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(D)) y^{p} = \sum_{s=0}^{n} \Delta_{(s)} (y+1)^{k}, \quad \text{if } D \text{ is ample,}$$

where 
$$\Delta_{(s)} := \sum_{\Gamma \in \mathcal{F}_s(\Delta)} l^*(\Gamma)$$
.

# 4. Relation with the original Bott formula and some combinatorial identities

In this section we compare our results of theorems 2.14 and 3.6 with the original Bott formula of Theorem 1.1. Recall that  $\mathcal{O}_{\mathbf{P}^n}(k)$  is ample if and only if k > 0, and the support polytope for  $\mathcal{O}_{\mathbf{P}^n}(k)$  is the simplex

$$\Delta = \Delta(k) = \{ m \in M_{\mathbf{R}} : m_1 \ge 0, \dots, m_n \ge 0, \quad m_1 + \dots + m_n \le k \}.$$

It is easy to see that

$$f_{n-j} = \binom{n+1}{j}, \quad \Delta^{(j)} = \binom{n+k-j}{k} \binom{n+1}{j},$$

and

$$f_s = {n+1 \choose n-s}, \quad \Delta_{(s)} = {k-1 \choose s} {n+1 \choose n-s},$$

for all  $0 \le j \le n$  and  $0 \le s \le n$ . Comparison of the formulas 1) of Theorems 2.14 and 1.1 corresponding to the case q = p gives the identity

$$\sum_{j=0}^{p} (-1)^{p-j} \binom{n-j}{p-j} \binom{n+1}{j} = 1.$$
 (a')

Now we compare the formulas 2) in these theorems corresponding to the case q=0:

$$\sum_{i=0}^{p} (-1)^{j} \binom{n-j}{p-j} \binom{n+k-j}{k} \binom{n+1}{j} = \binom{n+k-p}{k} \binom{k-1}{p}.$$
 (b')

Analoguous comparisons of Theorems 3.6 and 1.1 show that

$$\sum_{s=n}^{n} (-1)^{p+s} \binom{s}{p} \binom{n+1}{n-s} = 1, \tag{a"}$$

and

$$\sum_{s=p}^{n} \binom{s}{p} \binom{k-1}{s} \binom{n+1}{n-s} = \binom{n+k-p}{k} \binom{k-1}{p}. \tag{b"}$$

These identities can be proved directly.

Let us verify the simplest identity (a'). To prove (a') it is sufficient to check the functional equation

$$\sum_{j=0}^{n} {n+1 \choose j} t^{j} (1-t)^{n-j} = \sum_{k=0}^{n} t^{k},$$

and then to compare the coefficients by the monomial  $t^p$ . This equation follows from Corollary 2.15, but we can deduce it from the obvious identity

$$(t+1-t)^{n+1} = 1,$$

or

$$(1-t)\sum_{j=0}^{n} \binom{n+1}{j} t^{j} (1-t)^{n-j} + t^{n+1} = 1.$$

Now we get the required by  $(1 - t^{n+1})(1 - t)^{-1} = \sum_{k=0}^{n} t^{k}$ .

The identity (a'') is equivalent to (a'). The remaining identities (b') and (b'') have been proved in [MY]. Our proof is based on the method of integral representations for combinatorial sums in the spirit of the work [EY].

## 5. The Hilbert-Ehrhart Polynomials and the Serre Duality

Let  $\Delta \subset \mathbf{R}^n$  be a non-empty integer polytope. For a positive integer k, let  $k \cdot \Delta := \{kx : x \in \Delta\}$  denote the dilated polytope. It was proved by Ehrhart [E], and in somewhat stronger form by Macdonald [Mac], that there is a polynomial L(k) with the properties

(i) for any integer k > 0, one has

$$L(k) = l(k \cdot \Delta);$$

(ii) for any integer k > 0, one has the following reciprocity law

$$L(-k) = l^*(k \cdot \Delta).$$

The polynomial L(k) is called the *Ehrhart polynomial* for  $\Delta$ . In this section we study a generalization of the Ehrhart polynomial using the techniques of algebraic geometry (cf., [D1, O1]). The basic idea is to compare the Ehrhart polynomial with a certain Hilbert polynomial. For example, the Serre duality provides a generalization of the reciprocity law.

**Proposition 5.1** ([Sn, K]). Let  $\mathcal{L}$  be an invertible sheaf on a complete variety  $\mathbf{P}$  and  $\mathcal{F}$  be a coherent sheaf of  $\mathcal{O}_{\mathbf{P}}$ -modules on  $\mathbf{P}$ . Then the Euler-Poincaré characteristic  $\chi(\mathbf{P}, \mathcal{F} \otimes \mathcal{L}^{\otimes k})$  is a polynomial in k of total degree  $\leq n$  which assumes integer values whenever k is integer.

**Definition 5.2.** Let  $\mathcal{L} = \mathcal{O}_{\mathbf{P}}(D)$  be an invertible sheaf corresponding to the ample Cartier divisor D of a complete simplicial toric variety  $\mathbf{P}$ . We call the polynomial

$$L_p(k) := \chi(\mathbf{P}, \Omega_{\mathbf{P}}^p \otimes \mathcal{L}^{\otimes k}) = \chi(\mathbf{P}, \Omega_{\mathbf{P}}^p(kD))$$

the Hilbert polynomial of the sheaf  $\Omega^p_{\mathbf{P}}$  with respect to  $\mathcal{L}$ , or p-th Hilbert polynomial.

It follows from the Bott formula and Bott vanishing theorem for toric varieties, that

$$L_p(k) = h^0(\mathbf{P}, \Omega_{\mathbf{P}}^p(kD)) = \sum_{j=0}^p (-1)^j \binom{n-j}{p-j} \sum_{F \in \mathcal{F}_{n-j}(\Delta)} l(k \cdot F)$$
$$= \sum_{s=p}^n \binom{s}{p} \sum_{\Gamma \in \mathcal{F}_s(\Delta)} l^*(k \cdot \Gamma)$$

for k > 0. For example, for p = 0 the polynomial  $L_0(k)$  coincides with the usual Ehrhart polynomial of the polytope  $\Delta = \Delta(\mathcal{L})$ , and with  $L_n(k) = l^*(k \cdot \Delta)$  whenever k > 0. Hence, we can consider  $L_p(k)$  as a generalization of the Ehrhart polynomial and also call it the p-th Ehrhart polynomial for  $\Delta$ , or simply the p-th Hilber-Ehrhart polynomial.

Note that the reciprocity law can be written as

$$L_0(-k) = (-1)^n L_n(k).$$

Here we prove a more general result.

**Theorem 5.3.** The polynomial  $L_p(k)$  satisfies the duality property

$$L_n(-k) = (-1)^n L_{n-n}(k)$$

for any positive integer k and  $0 \le p \le n$ .

*Proof.* We need a special form of the Serre-Grothendieck duality (see §3.3 of [O1])

$$H^q(\mathbf{P}, \Omega^p_{\mathbf{P}} \otimes \mathcal{F})^* \simeq H^{n-q}(\mathbf{P}, \Omega^{n-p}_{\mathbf{P}} \otimes \mathcal{F}^*), \quad 0 \le q \le n$$

for any locally free sheaf of the  $\mathcal{O}_{\mathbf{P}}$ -modules  $\mathcal{F}$  with the dual  $\mathcal{F}^*$  on a complete simplicial toric variety  $\mathbf{P}$ . Take  $\mathcal{F} = \mathcal{O}_{\mathbf{P}}(kD)$ , k > 0. From the Serre duality we have the isomorphism

$$H^0(\mathbf{P}, \Omega^p_{\mathbf{P}}(kD))^* \simeq H^n(\mathbf{P}, \Omega^{n-p}_{\mathbf{P}}(-kD)).$$

Hence, using the vanishing of  $h^q(\mathbf{P}, \Omega^p_{\mathbf{P}}(kD))$  for q > 0, we get the equality

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(kD)) = (-1)^{n} \chi(\mathbf{P}, \Omega_{\mathbf{P}}^{n-p}(-kD)),$$

which is equivalent to

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(-kD)) = (-1)^{n} \chi(\mathbf{P}, \Omega_{\mathbf{P}}^{n-p}(kD)),$$

and this completes the proof.

Now let **P** be nonsingular. Recall, that any variety **P** has a Todd homology class  $td_{\mathbf{P}}$  of **P** in  $A_*(\mathbf{P})_{\mathbf{Q}}$ , see [F1]. Since **P** is nonsingular,  $td_{\mathbf{P}} = Td_{\mathbf{P}} \cap [\mathbf{P}]$ , where  $Td_{\mathbf{P}}$  is the Todd cohomogy class in  $A^*(\mathbf{P})_{\mathbf{Q}}$  and  $[\mathbf{P}]$  is the fundamental class of **P**.

We state without proof some facts on intersection theory on toric varieties, where part (a) slightly generalizes the results in [D1, F1, O1], and (b) obviously follows from the Hirzebruch-Riemann-Roch theorem [Hi].

**Proposition 5.4.** Let  $L_p(k)$  be the Hilbert-Ehrhart polynomial.

(a) The coefficients of the polynomial  $L_p(k) = \sum_{i=0}^n a_{pi} k^i$  are the intersection numbers

$$a_{pi} = \frac{1}{i!} \int_{\mathbf{P}} D^i \cdot ch(\Omega_{\mathbf{P}}^p) \cdot Td_{\mathbf{P}},$$

where  $\int_{\mathbf{P}}$  denotes the degree homomorphism  $\int_{\mathbf{P}} : A^0(\mathbf{P})_{\mathbf{Q}} \to \mathbf{Q}$  and  $ch(\Omega^p_{\mathbf{P}})$  is the Chern character of the sheaf  $\Omega^p_{\mathbf{P}}$ . For example, the leading coefficient is

$$a_{pn} = \binom{n}{p} Vol(\Delta),$$

where  $Vol(\Delta)$  is the normalized volume of the polytope  $\Delta \subset M_{\mathbf{R}}$  such that the volume of the unit cube determined by the basis of the lattice M is 1.

(b) The generating function for the polynomial  $L_p(k)$  is equal to

$$L(y;k) := \sum_{p=0}^{n} L_p(k) y^p = \int_{\mathbf{P}} e^{kD(y+1)} \prod_{j=1}^{d} Q(y; D_j),$$

where

$$Q(y;x) = \frac{x(y+1)}{e^{x(y+1)} - 1} + x,$$

and  $D_j$  are the torus-invariant divisors corresponding to the one-dimensional generators of the fan  $\Sigma$ .

# 6. Combinatorics of simple polytopes

Comparison of two versions of the Bott formula gives some elegant corollaries in combinatorics of simple polytopes.

If we compare the first parts of Theorems 2.14 and 3.6, we get the well-known Dehn-Sommerville equations, conjectured by Dehn [De] and proved by Sommerville [So]. (See also [Br].) In algebro-geometrical context these equalities were proved in [St] for any simple rational polytope, using the Poincaré duality on toric varities, and by Oda [O1], using the Serre-Grothendieck duality theorem.

**Theorem 6.1.** For any simple lattice n-polytope  $\Delta$  and  $p = 0, 1, \dots, n$  one has the identities

$$\sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} f_{n-j} = \sum_{s=p}^{n} (-1)^{s} \binom{s}{p} f_{s}.$$

Now compare the second parts of Theorems 2.14 and 3.6. We get non-trivial relations between the integer points in faces of the simple integer polytope  $\Delta$ , which we prove here in a purely combinatorial way.

**Theorem 6.2.** For any simple lattice n-polytope  $\Delta$  and p = 0, 1, ..., n one has the identities

$$\sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} \sum_{F \in \mathcal{F}_{n-j}(\Delta)} l(F) = \sum_{s=p}^{n} \binom{s}{p} \sum_{\Gamma \in \mathcal{F}_{s}(\Delta)} l^{*}(\Gamma).$$

*Proof.* We have the chain of equalities

$$\begin{split} \sum_{s=p}^{n} \binom{s}{p} \sum_{\Gamma \in \mathcal{F}_{s}(\Delta)} l^{*}(\Gamma) &= \sum_{\Gamma \subseteq \Delta} \binom{\dim \Gamma}{p} l^{*}(\Gamma) \\ &= \sum_{\Gamma \subseteq \Delta} \binom{\dim \Gamma}{p} \sum_{F \subseteq \Gamma} (-1)^{\dim \Gamma - \dim F} l(F) \\ &= \sum_{F \subseteq \Delta} (-1)^{\dim F} l(F) \sum_{\Gamma \supseteq F} (-1)^{\dim \Gamma} \binom{\dim \Gamma}{p} \\ &= \sum_{F \subseteq \Delta} (-1)^{n + \dim F} \binom{\dim F}{n - p} l(F) \\ &= \sum_{j=n-p}^{n} (-1)^{n+j} \binom{j}{n - p} \sum_{F \in \mathcal{F}_{s}(\Delta)} l(F) \\ &= \sum_{j=0}^{p} (-1)^{j} \binom{n - j}{p - j} \sum_{F \in \mathcal{F}_{s}(\Delta)} l(F), \end{split}$$

where we have used the identity

$$\sum_{\Gamma \supset F} (-1)^{\dim \Gamma} \binom{\dim \Gamma}{p} = (-1)^n \binom{\dim F}{n-p},$$

which is proved in the Appendix.

Remark 6.3. There are two obvious cases of the identities of Theorem 6.2. For p = 0, we have

$$l(\Delta) = \sum_{\Gamma \subset \Delta} l^*(\Gamma).$$

For p = n, we get the inclusion-exclusion formula

$$l^*(\Delta) = \sum_{j=0}^n (-1)^{n-j} \sum_{F \in \mathcal{F}_j(\Delta)} l(F).$$

We are in a position to give a purely combinatorial proof of the generalized reciprocity law of Theorem 5.3.

Corollary 6.4. For any simple lattice n-polytope  $\Delta$  and any positive integer k, one

$$L_p(-k) = (-1)^n L_{n-p}(k),$$

for any  $p = 0, \ldots, n$ .

*Proof.* Using the equality of Theorem 6.2, we get

$$L_{p}(-k) = \sum_{j=0}^{p} (-1)^{j} \binom{n-j}{p-j} \sum_{F \in \mathcal{F}_{n-j}(\Delta)} (-1)^{n-j} l^{*}(k \cdot F)$$

$$= (-1)^{n} \sum_{s=p}^{n} \binom{s}{n-p} \sum_{F \in \mathcal{F}_{s}(\Delta)} l^{*}(k \cdot F)$$

$$= (-1)^{n} L_{n-p}(k),$$

as required.

**Example 6.5.** Let  $\Delta$  be a convex polytope in  $\mathbb{R}^3$  with vertices at (0, 0, 0), (1, 0, 0), (0, 1, 0) and (1, 1, m), where m is a positive integer. Then

$$L_0(k) = \frac{m}{6}k^3 + k^2 + \frac{12 - m}{6}k + 1,$$

$$L_1(k) = \frac{m}{2}k^3 + k^2 - \frac{m}{2}k - 1,$$

$$L_2(k) = \frac{m}{2}k^3 - k^2 - \frac{m}{2}k + 1,$$

$$L_3(k) = \frac{m}{6}k^3 - k^2 + \frac{12 - m}{6}k - 1.$$

#### 7. Weighted components of sheaves with logarithmic poles

Recall that if  $\mathcal{L}$  is an ample invertible sheaf on a complete simplicial toric variety  $\mathbf{P}$ , then

$$H^q(\mathbf{P}, \Omega^p_{\mathbf{P}} \otimes \mathcal{L}) = 0$$

for all q > 0 and  $p \ge 0$  (see [D1]). This statement is a generalization of the theorems of Bott and Steenbrink for projective spaces and weighted projective spaces respectively. Batyrev and Cox proved a more general vanishing theorem.

**Definition 7.1** ([BC]). Denote by  $\Omega^p_{\mathbf{P}}(\log(-K))$  the sheaves of differential p-forms with logarithmic poles along the anticanonical divisor -K on  $\mathbf{P}$ . Let  $\mathcal{W}$  be the weight filtration

$$\mathcal{W}: \ 0 \subset W_0\Omega_{\mathbf{P}}^p(\log(-K)) \subset W_1\Omega_{\mathbf{P}}^p(\log(-K)) \subset \cdots \subset W_p\Omega_{\mathbf{P}}^p(\log(-K)) = \Omega_{\mathbf{P}}^p(\log(-K))$$
on  $\Omega_{\mathbf{P}}^p(\log(-K))$  defined by  $W_k\Omega_{\mathbf{P}}^p(\log(-K)) := \Omega_{\mathbf{P}}^{p-k} \wedge \Omega_{\mathbf{P}}^k(\log(-K))$ .

**Theorem 7.2** ([BC], Theorem 7.2). Let  $\mathcal{L}$  be an ample invertible sheaf on a complete simplicial toric variety  $\mathbf{P}$ . Then for any  $p \geq 0$ ,  $k \geq 0$ , and q > 0, one has

$$H^q(\mathbf{P}, W_k \Omega_{\mathbf{P}}^p(\log(-K)) \otimes \mathcal{L}) = 0.$$

Here we give some additional information about the global sections of  $W_k\Omega^p_{\mathbf{P}}(\log(-K))\otimes \mathcal{L}$ . We need the following result of V.I. Danilov.

**Theorem 7.3** ([D1], §15.7). For any integer  $0 \le k \le p$  one has the short exact sequence

$$(4) 0 \to W_{k-1}\Omega_{\mathbf{P}}^{p}(\log(-K)) \to W_{k}\Omega_{\mathbf{P}}^{p}(\log(-K)) \xrightarrow{\mathrm{Res}} \bigoplus_{\sigma \in \Sigma(k)} \Omega_{V(\sigma)}^{p-k} \to 0,$$

where  $V(\sigma)$  is the Zariski closure of the torus-invariant orbit of **P** corresponding to  $\sigma \in \Sigma(k)$  and Res is the Poincaré residue map.

It is easy to see that by tensoring by  $\mathcal{L}$  the short exact sequence (4) and applying the vanishing Theorem 7.2 we obtain

$$0 \to H^{0}(\mathbf{P}, W_{k-1}\Omega_{\mathbf{P}}^{p}(\log(-K)) \otimes \mathcal{L}) \to H^{0}(\mathbf{P}, W_{k}\Omega_{\mathbf{P}}^{p}(\log(-K)) \otimes \mathcal{L}) \to$$
$$\to \bigoplus_{\sigma \in \Sigma(k)} H^{0}(\mathbf{P}, \Omega_{V(\sigma)}^{p-k} \otimes \mathcal{L}) \to 0.$$

Thus, using an induction, we get

$$h^{0}(\mathbf{P}, W_{k}\Omega_{\mathbf{P}}^{p}(\log(-K)) \otimes \mathcal{L}) = \sum_{s=0}^{k} \sum_{\sigma \in \Sigma(s)} h^{0}(\mathbf{P}, \Omega_{V(\sigma)}^{p-s} \otimes \mathcal{L}).$$

The generalized Bott formula from Theorem 2.14 and Proposition 2.7 implies

$$h^{0}(\mathbf{P}, \Omega_{V(\sigma)}^{p-s} \otimes \mathcal{L}) = \sum_{j=0}^{p-s} (-1)^{j} \binom{n-s-j}{p-s-j} \sum_{\substack{\tau \in \Sigma(j) \\ \tau \succ \sigma}} l(\Delta_{\tau}).$$

We have proved the following statement.

**Proposition 7.4.** Let  $\mathbf{P}$  be a complete simplicial toric variety and  $\mathcal{L}$  be an ample invertible sheaf on  $\mathbf{P}$  with the support polytope  $\Delta$ . Then one has the equality

$$h^{0}(\mathbf{P}, W_{k}\Omega_{\mathbf{P}}^{p}(\log(-K)) \otimes \mathcal{L}) = \sum_{s=0}^{k} \sum_{\sigma \in \Sigma(s)} \sum_{j=0}^{p-s} (-1)^{j} \binom{n-s-j}{p-s-j} \sum_{\substack{\tau \in \Sigma(j) \\ \tau \succ \sigma}} l(\Delta_{\tau}).$$

## 8. Cohomology of quasi-smooth hypersurfaces

In this section we reprove the "combinatorial part" of the well-known result of Danilov and Khovanskii on the Hodge numbers of quasi-smooth hypersurfaces in toric varieties. Our proof is simple and uses the Bott formula for toric varieties.

Any complete simplicial toric variety  $\mathbf{P} = \mathbf{P}(\Sigma)$  can be constructed as a geometric quotient as follows (see [C]). Suppose that  $S(\Sigma) := \mathbf{C}[z_1, \ldots, z_d]$  is the polynomial ring over  $\mathbf{C}$  with the variables  $z_i$  corresponding to the integer generators  $v_i$  of the one-dimensional cones of  $\Sigma$ . For every  $\sigma \in \Sigma$  let  $\hat{z}_{\sigma} := \prod_{v_i \notin \sigma} z_i$ , and let

$$Z(\Sigma) := \bigcap_{\sigma \in \Sigma} \{ z \in \mathbf{C}^d : \, \hat{z}_{\sigma} = 0 \}.$$

The toric variety  $\mathbf{P}$  is a geometric quotient of the Zariski open set  $U(\Sigma) := \mathbf{C}^d \setminus Z(\Sigma)$  by the algebraic group  $G(\Sigma) := \operatorname{Hom}_{\mathbf{Z}}(A_{n-1}(\mathbf{P}), \mathbf{C}^*)$ , where  $A_{n-1}(\mathbf{P})$  is the Chow group of  $\mathbf{P}$ . The action of  $G(\Sigma)$  on  $\mathbf{C}^d$  induces the grading on  $S(\Sigma)$ . Moreover, if  $\mathcal{L}$  is an invertible sheaf on  $\mathbf{P}$ , then for  $\alpha = [\mathcal{L}] \in A_{n-1}(\mathbf{P})$  one has the isomorphism  $H^0(\mathbf{P}, \mathcal{L}) \simeq S_{\alpha}$ . A polynomial f in the graded part  $S_{\alpha}$  of  $S(\Sigma)$  corresponding to the class  $\alpha \in A_{n-1}(\mathbf{P})$  is said to be G-homogeneous of degree  $\alpha$ . Thus, the global sections of  $\mathcal{L}$  determine the Cartier divisor D on  $\mathbf{P}$ .

**Definition 8.1** ([BC]). The hypersurface  $D \subset \mathbf{P}$  defined by the G-homogeneous polynomial f in  $S(\Sigma)$  is said to be *quasi-smooth* if  $\mathbf{V}(f) \cap U(\Sigma)$  is either empty or a nonsingular subvariety of codimension one in  $U(\Sigma)$ .

**Definition 8.2.** Let X be an algebraic variety over  $\mathbb{C}$ . Denote by  $h^{p,\,q}(H^k(X,\mathbb{C}))$  the dimension of the  $(p,\,q)$ -component of the k-th cohomology group. Let us introduce the numbers

$$e^{p, q}(X) := \sum_{k} (-1)^k h^{p, q}(H^k(X, \mathbf{C})),$$
  
 $e^p(X) := \sum_{q} e^{p, q}(X).$ 

For a complete nonsingular variety X, we have

$$e^{p, q}(X) = (-1)^{p+q} h^{p, q}(X),$$

where  $h^{p, q}(X)$  are the (ordinary) Hodge numbers of X.

Remark 8.3. Danilov and Khovanskii use cohomologies with compact supports instead of usual cohomologies, but in our situation there exists the Poincaré duality

(5) 
$$h^{p,q}(H^k(X, \mathbf{C})) = h^{r-p, r-q}(H_c^{2r-k}(X, \mathbf{C})),$$

where  $r = \dim_{\mathbf{C}} X$ , which relates the two treatments.

Let  $D \subset \mathbf{P}$  be a nondegenerate divisor on a complete simplicial *n*-dimensional toric variety. Then the natural map

$$H^i(\mathbf{P}) \to H^i(D)$$

is an isomorphism for i < n-1 and is injective for i = n-1.

**Definition 8.4.** Define the primitive cohomology  $H_0^{n-1}(D)$  by the exact sequence

$$0 \to H^{n-1}(\mathbf{P}) \to H^{n-1}(D) \to H_0^{n-1}(D) \to 0.$$

Remark 8.5. As A. Mavlyutov pointed out to us (see [M2], Remark 5.5), the primitive cohomologies  $H_0^{n-1}(D)$  coincide with the residue part  $H_{res}^{n-1}(D)$  of cohomology defined as the image of the residue map  $Res: H^n(\mathbf{P}\backslash D) \to H^{n-1}(D)$ .

Recall, that since **P** is simplicial and D is nondegenerate,  $H_0^{n-1}(D)$  has a pure Hodge structure (cf., [BC]).

**Theorem 8.6** ([DK]). Let  $\Delta$  be a support polytope, corresponding to an ample nondegenerate hypersurface D. Then,

$$h_0^{p,\,n-1-p}(D)=(-1)^n\sum_{\Gamma\subseteq\Delta}\varphi_{\dim\Gamma-p}(\Gamma),$$

where

$$\varphi_i(\Gamma) := \sum_{k=1}^i (-1)^{i-k} {\dim \Gamma + 1 \choose i-k} l^*(k \cdot \Gamma).$$

Denote by  $\Omega_{\mathbf{P}}^p(\log D)$  the sheaf of p-differential forms on  $\mathbf{P}$  with logarithmic poles along D (see §15 of [D1]). First, we compute the Euler-Poincaré characteristic of the sheaf  $\Omega_{\mathbf{P}}^p(\log D)$ .

**Proposition 8.7** ([BC], Proposition 10.1). If D is a quasi-smooth hypersurface of a complete toric variety  $\mathbf{P}$  defined by zeros of a global section of the ample invertible sheaf  $\mathcal{O}_{\mathbf{P}}(D)$  on  $\mathbf{P}$ , then there is an exact sequence

$$0 \to \Omega_{\mathbf{P}}^{p}(\log D) \to \Omega_{\mathbf{P}}^{p}(D) \xrightarrow{d} \Omega_{\mathbf{P}}^{p+1}(2D)/\Omega_{\mathbf{P}}^{p+1}(D) \xrightarrow{d} \dots$$
$$\dots \xrightarrow{d} \Omega_{\mathbf{P}}^{n}((n-p+1)D)/\Omega_{\mathbf{P}}^{n}((n-p)D) \to 0.$$

**Lemma 8.8.** Let  $\Delta$  be the support polytope for  $\mathcal{O}_{\mathbf{P}}(D)$  corresponding to the ample divisor D. Then the Euler-Poincaré characteristics of the sheaf  $\Omega^p_{\mathbf{P}}(\log D)$  is equal to

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^p(\log D)) = \sum_{k=1}^{n-p} (-1)^{k+1} \sum_{\Gamma \subset \Delta} \binom{\dim \Gamma + 1}{p+k} l^*(k \cdot \Gamma).$$

*Proof.* Indeed, from the long exact sequence of Proposition 8.7, we have

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(\log D)) = \sum_{k=1}^{n-p} (-1)^{k+1} [\chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p+k-1}(kD)) + \chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p+k}(kD))].$$

Since D is ample,

$$\chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p}(\log D)) = \sum_{k=1}^{n-p} (-1)^{k+1} [h^{0}(\mathbf{P}, \Omega_{\mathbf{P}}^{p+k-1}(kD)) + h^{0}(\mathbf{P}, \Omega_{\mathbf{P}}^{p+k}(kD))],$$

applying the generalized Bott formula given in Theorem 3.6, we prove the statement.  $\hfill\Box$ 

We start with the relation

$$e^p(D) = e^{p+1}(\mathbf{P}) - e^{p+1}(\mathbf{P} \setminus D),$$

following from the Gysin exact sequence

$$\ldots \to H^{k-2}(D) \to H^k(\mathbf{P}) \to H^k(\mathbf{P}\backslash D) \to H^{k-1}(D) \to \ldots$$

of hypercohomologies of the exact sequence of complexes

$$0 \to \Omega_{\mathbf{P}}^{\cdot} \to \Omega_{\mathbf{P}}^{\cdot}(\log D) \to \Omega_D^{\cdot-1} \to 0.$$

There are two spectral sequences

$${}^{\prime}E_{1}^{p, q} = H^{q}(\mathbf{P}, \Omega_{\mathbf{P}}^{p}) \Rightarrow H^{p+q}(\mathbf{P}, \mathbf{C}),$$

and

$$^{\prime\prime}E_1^{p,\,q} = H^q(\mathbf{P}, \Omega_{\mathbf{P}}^p(\log D)) \Rightarrow H^{p+q}(\mathbf{P} \backslash D, \mathbf{C}),$$

both degenerating at the first term and converging to the Hodge filtration  $\mathcal{F}$  on  $H^*(\mathbf{P}, \mathbf{C})$  and  $H^*(\mathbf{P} \setminus D, \mathbf{C})$  respectively [D1]. Therefore, we have the equalities

$$e^{p+1}(\mathbf{P}) = (-1)^{p+1} \chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p+1}), \quad e^{p+1}(\mathbf{P} \backslash D) = (-1)^{p+1} \chi(\mathbf{P}, \Omega_{\mathbf{P}}^{p+1}(\log D)).$$

Using the Bott formula, one obtains:

Proposition 8.9. One has the equality

$$e^{p}(D) = (-1)^{p+1} \sum_{k \ge 0} (-1)^{k} {k \choose p+1} f_k - \sum_{\Gamma \subseteq \Delta} (-1)^{\dim \Gamma} \varphi_{\dim \Gamma - p}(\Gamma).$$

By applying the Poincaré duality (5), we restore all the Hodge numbers  $h_0^{p, q}(D)$  and get the statement of Theorem 8.6.

## 9. The Bott formula on $\mathbf{P}(\mathcal{E})$

Let  $\mathcal{L}_0, \ldots, \mathcal{L}_s$  be ample invertible sheaves on a complete *n*-dimensional toric variety  $\mathbf{P}$ . Denote  $Y = \mathbf{P}(\mathcal{E})$  the projective space bundle associated with the sheaf  $\mathcal{E} := \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_s$ , with the invertible sheaf  $\mathcal{O}_Y(1)$  and the projection  $\pi : Y \to \mathbf{P}$ . The sheaf  $\mathcal{O}_Y(1)$  is ample since the  $\mathcal{L}_j$  are ample (see §1 of [Ha]).

**Definition 9.1.** Let  $\Omega^1_{Y/\mathbf{P}}$  be the sheaf of relative differentials arising from the short exact sequence (see [Man])

(6) 
$$0 \to \Omega^1_{Y/\mathbf{P}} \to \pi^*(\mathcal{E}) \otimes \mathcal{O}_Y(-1) \to \mathcal{O}_Y \to 0.$$

The purpose of this section is to give a combinatorial description of the global sections of the sheaf of relative p-th differentials  $\Omega^p_{Y/\mathbf{P}}(k) := \bigwedge^p \Omega^1_{Y/\mathbf{P}} \otimes \mathcal{O}_Y(k)$ .

Recall the construction of Y as a toric variety (see [O1]). Suppose that the support polytope associated with the sheaf  $\mathcal{L}_j$  has the form

$$\Delta_j = \{ m \in M_{\mathbf{R}} : \langle m, v_i \rangle \ge -a_{ij}, \ i = 1, \dots, d \}.$$

Let  $N' \simeq \mathbf{Z}^s$  be a lattice with **Z**-basis  $\{e_1, \ldots, e_s\}$  and  $\widetilde{N} := N \oplus N'$ . The 1-dimensional cones of the fan  $\widetilde{\Sigma}$  corresponding to the toric variety Y have the generators

$$\widetilde{v}_{i} = v_{i} + \sum_{j=1}^{s} (a_{ij} - a_{i0})e_{j}, \quad i = 1, \dots, d,$$
 $\widetilde{e}_{0} = -e_{1} - \dots - e_{s},$ 
 $\widetilde{e}_{j} = e_{j}, \quad j = 1, \dots, s.$ 

Denote by  $\widetilde{\sigma}$  the image of each  $\sigma \in \Sigma$  under the map  $N_{\mathbf{R}} \to \widetilde{N}_{\mathbf{R}}$  given by  $v_i \mapsto \widetilde{v}_i$  and let  $\sigma'$  be the cones generated by  $e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_s$ . Then the fan  $\widetilde{\Sigma}$  is generated by the cones  $\widetilde{\sigma} + \sigma'$  and their faces. Let  $\eta_1, \ldots, \eta_s$  be the generators of the lattice M' dual to N'. Denote the torus invariant divisors on Y corresponding to  $\widetilde{v}_i$  and  $\widetilde{e}_j$  by  $\widetilde{D}_i$  and  $\widetilde{D}'_j$  respectively. Note that  $\pi^*(\mathcal{L}_j) = \mathcal{O}_Y(\sum_{i=1}^d a_{ij}\widetilde{D}_i)$ .

Lemma 9.2. The polytope

$$\nabla = \{\lambda_1 \eta_1 + \ldots + \lambda_s \eta_s + \lambda_0 m_0 + \ldots + \lambda_s m_s : \lambda_j \ge -k_j, \sum_{j=0}^s \lambda_j = k, m_j \in \Delta_j\}$$

in  $\widetilde{M}_{\mathbf{R}} := M'_{\mathbf{R}} \oplus M_{\mathbf{R}}$  is the support polytope associated with the sheaf  $\mathcal{O}_Y(k) \otimes \mathcal{O}_Y(k_0 \widetilde{D}'_0 + \ldots + k_s \widetilde{D}'_s)$ .

*Proof.* It follows from the Lemma 2.1 in [M1] that

$$\mathcal{O}_Y(1) \simeq \mathcal{O}_Y(\widetilde{D}_0') \otimes \pi^*(\mathcal{L}_0) = \mathcal{O}_Y(\widetilde{D}_0' + \sum_{i=1}^d a_{i0}\widetilde{D}_i)$$

and

$$\mathcal{O}_Y(k) \otimes \mathcal{O}_Y(k_0 \widetilde{D}'_0 + \ldots + k_s \widetilde{D}'_s) = \mathcal{O}_Y((k+k_0) \widetilde{D}'_0 + \sum_{i=1}^s k_j \widetilde{D}'_j + \sum_{i=1}^d k a_{i0} \widetilde{D}_i).$$

Each  $\widetilde{m} \in \widetilde{M}_{\mathbf{R}}$  can be uniquely written as  $\widetilde{m} = m + \lambda_1 \eta_1 + \ldots + \lambda_s \eta_s$ , where  $m \in M_{\mathbf{R}}, \lambda_j \in \mathbf{R}$ . The support polytope  $\nabla$  is defined by the inequalities

$$\langle \widetilde{m}, \widetilde{v}_i \rangle \ge -ka_{i0}, \quad i = 1, \dots, d,$$
  
 $\langle \widetilde{m}, \widetilde{e}_0 \rangle \ge -k - k_0,$   
 $\langle \widetilde{m}, \widetilde{e}_j \rangle \ge -k_j, \quad j = 1, \dots, s$ 

in  $\widetilde{M}_{\mathbf{R}}$ . Let  $\lambda_0 = k - \sum_{j=1}^s \lambda_j$ . Then the inequalities above are equivalent to

$$\{\langle m, v_i \rangle \ge -\sum_{j=0}^s \lambda_j a_{ij}, \quad i = 1, \dots, d, \quad \lambda_j \ge -k_j, \quad j = 0, \dots, s, \quad \sum_{j=0}^s \lambda_j = k\}.$$

The same arguments as in §3 of [CCD] show that  $m = \sum_{j=0}^{s} \lambda_j m_j$ , where  $m_j \in \Delta_j$ , and  $\widetilde{m} \in \nabla$  follows immediately.

**Definition 9.3.** For any  $J = \{j_1, \dots, j_t\} \subset I := \{0, \dots, s\}$  define the polytope

$$\Delta_{j_1...j_t}(k) := \{ \sum_{i=1}^s \lambda_i \eta_i + \sum_{t=0}^s \lambda_t m_t : \lambda_j \ge 1, \ j \in J, \ \lambda_j \ge 0, \ j \in I \setminus J, \ \sum_{j=0}^s \lambda_j = k, \ m_t \in \Delta_t \}.$$

**Theorem 9.4.** For any sufficiently large integer k > 0

$$h^{q}(Y, \Omega_{Y/\mathbf{P}}^{p}(k)) = \begin{cases} \sum_{t=0}^{p} (-1)^{p-t} \sum_{0 \le j_{1} < \dots < j_{t} \le s} l(\Delta_{j_{1}\dots j_{t}}(k)), & q = 0, \\ 0, & q > 0. \end{cases}$$

*Proof.* From the p-th exterior power of the sequence (6), we have

$$0 \to \Omega^p_{Y/\mathbf{P}}(k) \to \bigwedge^p \pi^*(\mathcal{E}) \otimes \mathcal{O}_Y(k-p) \to \Omega^{p-1}_{Y/\mathbf{P}}(k) \to 0.$$

Since  $\mathcal{O}_{Y}(1)$  is ample, for all  $k \gg 0$ , we obtain the exact sequence of global sections

$$0 \to H^0(Y, \Omega^p_{Y/\mathbf{P}}(k)) \to H^0(Y, \bigwedge^p \pi^*(\mathcal{E}) \otimes \mathcal{O}_Y(k-p)) \to H^0(Y, \Omega^{p-1}_{Y/\mathbf{P}}(k)) \to 0.$$

Lemma 2.1 of [M1] implies the isomorphisms  $\pi^*(\mathcal{L}_{j_m}) \otimes \mathcal{O}_Y(-1) \simeq \mathcal{O}_Y(-\widetilde{D}'_{j_m})$  and consequently, the isomorphisms

$$\bigwedge^{p} \pi^{*}(\mathcal{E}) \otimes \mathcal{O}_{Y}(k-p) \simeq \mathcal{O}_{Y}(k) \otimes \bigoplus_{0 \leq j_{1} < \dots < j_{p} \leq s} \pi^{*}(\mathcal{L}_{j_{1}}) \otimes \dots \otimes \pi^{*}(\mathcal{L}_{j_{p}}) \otimes \mathcal{O}_{Y}(-p)$$

$$\simeq \mathcal{O}_{Y}(k) \otimes \bigoplus_{0 \leq j_{1} < \dots < j_{p} \leq s} \mathcal{O}_{Y}(-\widetilde{D}'_{j_{1}} - \dots - \widetilde{D}'_{j_{p}}).$$

Hence we obtain the relations

$$h^{0}(Y, \Omega_{Y/\mathbf{P}}^{p}(k)) = \sum_{0 \leq j_{1} < \dots < j_{p} \leq s} l(\Delta_{j_{1} \dots j_{p}}(k)) - h^{0}(Y, \Omega_{Y/\mathbf{P}}^{p-1}(k)).$$

By induction we get the requested formula.

## 10. Appendix

In the proof of Theorem 6.2 we have used the following identity.

**Lemma 10.1.** For any face F of a simple n-polytope  $\Delta$  there are relations

$$\sum_{\Gamma\supset F} (-1)^{\dim\Gamma} \binom{\dim\Gamma}{p} = (-1)^n \binom{\dim F}{n-p}.$$

*Proof.* Since  $\Delta$  is a simple, there are precisely

$$\binom{n-s}{n-j}$$

*j*-faces of  $\Delta$  containing a given s-face F of  $\Delta$ . Hence, the requested formula is equivalent to the combinatorial identity

$$\sum_{j=0}^{n} (-1)^{j} {j \choose p} {n-s \choose n-j} = (-1)^{n} {s \choose n-p},$$

or

$$\sum_{i=0}^{n-s} (-1)^j \binom{n-j}{p} \binom{n-s}{j} = \binom{s}{n-p}.$$

We give a proof of this identity based on the method of integral representations. Rewrite the sum

$$S = \sum_{j=0}^{n-s} (-1)^j \binom{n-j}{p} \binom{n-s}{j}$$

as

$$S = \sum_{i=0}^{\infty} (-1)^{j} \frac{1}{(2\pi i)^{2}} \iint_{\gamma} \frac{(1+z)^{n-j} (1+w)^{n-s}}{z^{p+1} w^{j+1}} dz dw,$$

where the cycle  $\gamma$  is  $\{|z|=\varepsilon\} \times \{|w|=\delta\}$ . One can choose the numbers  $\varepsilon>0$  and  $\delta>0$  small enough, say  $\varepsilon=1/2$ ,  $\delta=3$ , so that the geometric series  $\sum_{j=0}^{\infty}(-(1+z)^{-1}w^{-1})^j$  converges on the contour of integration  $\gamma$ , and it is possible to reverse the order of integration and summation. Therefore, summing up a geometric sequence,

$$S = \frac{1}{(2\pi i)^2} \iint_{\gamma} \frac{(1+z)^n (1+w)^{n-s}}{z^{p+1} w} \sum_{j=0}^{\infty} \left(\frac{-1}{(1+z)w}\right)^j dz dw$$

$$= \frac{1}{(2\pi i)^2} \iint_{\gamma} \frac{(1+z)^n (1+w)^{n-s}}{z^{p+1} w \left(1 + \frac{1}{(1+z)w}\right)} dz dw$$

$$= \frac{1}{2\pi i} \int_{|z|=1/2} \frac{(1+z)^{n+1}}{z^{p+1}} dz \cdot \frac{1}{2\pi i} \int_{|w|=3} \frac{(1+w)^{n-s}}{1+zw+w} dw.$$

By the residue theorem, the second integral is

$$\frac{1}{2\pi i} \int_{|w|=3} \frac{(1+w)^{n-s}}{1+zw+w} dw = res_{w=w_0} \frac{(1+w)^{n-s}}{(1+z)(w+1/(1+z))}$$

$$= \frac{\left(1 - \frac{1}{1+z}\right)^{n-s}}{1+z} = \frac{z^{n-s}}{(1+z)^{n-s+1}},$$

where  $w_0 = -1/(1+z) \in \{|w| = 3\}$ , since  $|w_0| \le \frac{1}{1-1/2} = 2 < 3$  for |z| = 1/2. Finally,

$$S = \frac{1}{2\pi i} \int_{|z|=1/2} \frac{(1+z)^{n+1}}{z^{p+1}} \cdot \frac{z^{n-s}}{(1+z)^{n-s+1}} dz$$
$$= \frac{1}{2\pi i} \int_{|z|=1/2} \frac{(1+z)^s}{z^{p-n+s+1}} dz = \binom{s}{p-n+s} = \binom{s}{n-p},$$

which concludes the proof.

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